Improving the Modified Gauss-Seidel Method for Z-Matrices

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ABSTRACT

In 1991 A. D. Gunawardena et al. reported that the convergence rate of the Gauss-Seidel method with a preconditioning matrix \( I + S \) is superior to that of the basic iterative method. In this paper, we use the preconditioning matrix \( I + S(\alpha) \). If a coefficient matrix \( A \) is an irreducibly diagonally dominant Z-matrix, then \( [I + S(\alpha)]A \) is also a strictly diagonally dominant Z-matrix. It is shown that the proposed method is also superior to other iterative methods. © 1997 Elsevier Science Inc.

1. INTRODUCTION

Let us consider iterative methods for the solution of the linear system

\[
Ax = b,
\]

where \( A \) is an \( n \times n \) square matrix, and \( x \) and \( b \) are vectors. Then the basic iterative scheme for Equation (1) is

\[
Mx_{k+1} = Nx_k + b, \quad k = 0, 1, \ldots,
\]
where \( A = M - N \), and \( M \) is nonsingular. Thus (2) can also be written as

\[
x_{k+1} = T x_k + c, \quad k = 0, 1, \ldots,
\]

where \( T = M^{-1}N \), \( c = M^{-1}b \). Assuming \( A = I - L - U \), where \( I \) is the identity matrix, and \( L \) and \( U \) are strictly lower and strictly upper triangular matrices, respectively, the iteration matrix of the classical Gauss-Seidel method is given by \( T = (I - L)^{-1}U \).

We now transform the original system (1) into the preconditioned form

\[
P A x = P b.
\]

Then, we can define the basic iterative scheme:

\[
M_p x_{k+1} = N_p x_k + P b, \quad k = 0, 1, \ldots
\]

where \( P A = M_p - N_p \) and \( M_p \) is nonsingular.

Recently, Gunawardena et al. [1] proposed the modified Gauss-Seidel method with \( P = I + S \), where

\[
S = \begin{pmatrix}
0 & -a_{12} & 0 & \cdots & 0 \\
0 & 0 & -a_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -a_{n-1,n} \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The performance of this method on some matrices is investigated in [1].

In this paper, we propose a scheme for improving of the modified Gauss-Seidel method and discuss convergence. Finally, we show that this method yields a considerable improvement in the rate of convergence.

2. PROPOSED METHOD

First, let us summarize the modified Gauss-Seidel method [1] with the preconditioner \( P = I + S \). Let all elements \( a_{ii+1} \) of \( S \) be nonzero. Then we have

\[
\tilde{A} x = (I + S) A x = \left[ I - L - SL - (U - S + SU) \right] x,
\]

\[
\tilde{b} = (I + S) b.
\]
Whenever
\[ a_{ii+1}a_{i+1i} \neq 1 \quad \text{for} \quad i = 1, 2, \ldots, n - 1, \]
\((I - SL - L)^{-1}\) exists, and hence it is possible to define the Gauss-Seidel iteration matrix for \(\tilde{A}\), namely
\[
\tilde{T} = (I - SL - L)^{-1}(U - S + SU).
\] (7)

This iteration matrix \(\tilde{T}\) is called the modified Gauss-Seidel iteration matrix. We next propose a new iterative method with the preconditioned matrix,
\[ P = I + S(\alpha), \]
where \(S(\alpha)\) is
\[
S(\alpha) = \begin{pmatrix}
0 & -\alpha_1 a_{12} & 0 & \cdots & 0 \\
0 & 0 & -\alpha_2 a_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -\alpha_{n-1} a_{n-1n} \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]

Thus we obtain
\[
A(\alpha) = [I + S(\alpha)]A = I - L - S(\alpha)L - [U - S(\alpha) + S(\alpha)U],
\]
\[
b(\alpha) = [I + S(\alpha)]b.
\] (8)

Whenever
\[ \alpha_i a_{ii+1}a_{i+1i} \neq 1 \quad \text{for} \quad i = 1, 2, \ldots, n - 1, \]
\([I - S(\alpha)L - L]^{-1}\) exists, and hence it is possible to define the Gauss-Seidel iteration matrix for \(A(\alpha)\), namely
\[
T(\alpha) = [I - S(\alpha)L - L]^{-1}[U - S(\alpha) + S(\alpha)U].
\] (9)

**Remark 1.** In (9), if \(\alpha_i = 0\) for all \(i\), \(T(\alpha)\) reduces to the classical Gauss-Seidel iterative method, and if \(\alpha_i = 1\) for all \(i\), \(T(\alpha)\) reduces to the modified Gauss-Seidel iterative method.
3. CONVERGENCE OF THE PROPOSED METHOD

First, we give a well-known result [2, 3].

**Lemma 2.** An upper bound on the spectral radius \( \rho(T) \) for the Gauss-Seidel iteration matrix \( T \) is given by

\[
\rho(T) \leq \max_i \frac{\tilde{u}_i}{1 - \tilde{l}_i},
\]

where \( \tilde{l}_i \) and \( \tilde{u}_i \) are the sums of the moduli of the elements in row \( i \) of the triangular matrices \( L \) and \( U \), respectively.

Next, we discuss the convergence of the proposed method. Let \( A(\alpha) = D(\alpha) - E(\alpha) - F(\alpha) \), where \( D(\alpha) \), \( -E(\alpha) \), and \( -F(\alpha) \) are the diagonal, strictly lower triangular, and strictly upper triangular parts of \( A(\alpha) \). Then the elements of \( A(\alpha) \) are

\[
\bar{a}_{ij} = \begin{cases} 
  a_{ij} - \alpha_1 a_{i+1,j} a_{i+1,j}, & 1 \leq i < n, \\
  a_{nj}, & i = n
\end{cases}
\]

(10)

If \( A \) is a diagonally dominant \( Z \)-matrix, then we have

\[
0 \leq a_{i,i+1} a_{i+1,j} \leq 1 \quad \text{for} \quad j \neq i + 1,
\]

\[
-1 \leq a_{i,i+1} a_{i+1,i+1} \leq 0.
\]

(11)

Therefore, the following inequalities hold:

\[
a_{i,i+1} a_{i+1,i} \geq 0
\]

\[
a_{i,i+1} \sum_{j=1}^{i-1} a_{i+1,j} \geq 0,
\]

\[
a_{i,i+1} \sum_{j=i+1}^{n} a_{i+1,j} \leq 0, \quad 1 \leq i < n.
\]
For simplicity we denote

\[ p_i = a_{i+1}a_{i+1} \]
\[ q_i = a_{i+1} \sum_{j=1}^{i-1} a_{i+1,j} \]
\[ r_i = a_{i+1} \sum_{j=i+1}^{n} a_{i+1,j} \quad \text{for } 1 \leq i < n, \]

and set

\[ p_n = 0, \]
\[ q_n = 0, \]
\[ r_n = 0. \]

Then the following inequality holds:

\[ p_i + q_i + r_i = a_{i+1} \sum_{j=1}^{n} a_{i+1,j} \leq 0, \quad 1 \leq i < n. \]

Furthermore, if \( a_{i+1} \neq 0 \) and \( \sum_{j=1}^{n} a_{i+1,j} < 0 \) for some \( i < n \), then we have

\[ p_i + q_i + r_i < 0 \quad \text{for some } i < n. \quad (12) \]

**Theorem 3.** Let \( A \) be a nonsingular diagonally dominant Z-matrix with unit diagonal elements and \( \sum_{j=1}^{n} a_{ij} > 0 \). Assume that \( \sum_{j=1}^{n} a_{i+1,j} > 0 \) if \( \sum_{j=1}^{n} a_{ij} = 0 \) for some \( i < n \). Then \( A(\alpha) \) is a strictly diagonally dominant Z-matrix, and \( \rho(T(\alpha)) < 1 \) for \( 0 \leq \alpha_i \leq 1 \) (1 \leq i < n).

**Proof.** Let \( d(\alpha)_i \), \( l(\alpha)_i \), and \( u(\alpha)_i \) be the sums of the elements in row \( i \) of \( D(\alpha) \), \( L(\alpha) \), and \( U(\alpha) \), respectively. The following equations hold:

\[ d(\alpha)_i = \bar{a}_{ij} = 1 - \alpha_i p_i, \quad 1 \leq i \leq n, \]
\[ l(\alpha)_i = - \sum_{j=1}^{i-1} \{ \bar{a}_{ij} \} = l_i + \alpha_i q_i, \quad 1 \leq i \leq n, \quad (13) \]
\[ u(\alpha)_i = - \sum_{j=i+1}^{n} \{ \bar{a}_{ij} \} = u_i + \alpha_i r_i, \quad 1 \leq i \leq n, \]
where $l_i$ and $u_i$ are the sums of the elements in row $i$ of $L$ and $U$ for $A = I - L - U$, respectively. Since $A$ is a diagonally dominant Z-matrix, from (11) the following relations hold:

$$1 - \alpha_i a_{i,i+1} a_{i+1,j} > 0 \quad \text{for } j = i,$$

$$a_{ij} - \alpha_i a_{i,i+1} \sum_{k=1}^{i-1} a_{i+1,k} \leq 0 \quad \text{for } i > j,$$

$$(1 - \alpha_i) a_{ij} - \alpha_i a_{i,i+1} \sum_{k=i+2}^{n} a_{i+1,k} \leq 0 \quad \text{for } i < j.$$ 

Therefore, $l(\alpha)_i > 0$, $u(\alpha)_i > 0$, and $A(\alpha)$ is a Z-matrix. Moreover, from (12) and the assumption, we can easily obtain

$$d(\alpha)_i - l(\alpha)_i - u(\alpha)_i = (d_i - l_i - u_i) - \alpha_i (p_i + q_i + r_i) > 0$$

for all $i$. (14)

Therefore, $A(\alpha)$ satisfies the condition of diagonal dominance. From $u(\alpha)_i \geq 0$, we have

$$d(\alpha)_i - l(\alpha)_i > u(\alpha)_i \geq 0 \quad \text{for all } i.$$ 

This implies

$$\frac{u(\alpha)_i}{d(\alpha)_i - l(\alpha)_i} < 1.$$ (15)

Hence, $\rho(T(\alpha)) < 1$, by Lemma 3.

THEOREM 4. Let $A$ be a matrix satisfying the conditions in Theorem 3. Put $\alpha'_i = (1 - l_i - u_i - 2a_{i,i+1})/(p_i + q_i + r_i - 2a_{i,i+1})$ for $1 \leq i < n$. Then $\alpha'_i > 1$, $A(\alpha)$ is a strictly diagonally dominant matrix, and $\rho(T(\alpha)) < 1$ for $1 \leq \alpha < \alpha'_i$.\[\Box\]
**Proof.** Since $\sum_{j=1, j \neq i}^{n} a_{i+1,j} < 0$, we have

\[
p_i + q_i + r_i - 2a_{i+1} = a_{i+1} \left( \sum_{j=1}^{n} a_{i+1,j} - 2 \right) = a_{i+1} \left( \sum_{j=1}^{n} a_{i+1,j} - 1 \right) > 0 \quad \text{for } 1 \leq i < n, \tag{16}
\]

and

\[
1 - l_i - u_i - 2a_{i+1,j} > p_i + q_i + r_i - 2a_{i+1} > 0 \quad \text{for } 1 \leq i < n,
\]

since $p_i + q_i + r_i < 0$. This implies

\[
\frac{1 - l_i - u_i - 2a_{i+1}}{p_i + q_i + r_i - 2a_{i+1}} > 1 \quad \text{for } 1 \leq i < n.
\]

That is, $\alpha'_i > 1$ for $1 \leq i < n$. Let

\[
\bar{u}(\alpha)_i = \sum_{j=i+1}^{n} |a_{ij} - \alpha a_{i+1,a_i+1,j}| \quad \text{for } 1 \leq i < n.
\]

Then for $\alpha_i > 1$ ($1 \leq i < n$) the following relation holds:

\[
\bar{u}(\alpha)_i = |(1 - \alpha_i)a_{i+1}i + \sum_{j=i+2}^{n} |a_{ij} - \alpha a_{i+1,a_i+1,j}|
\]

\[
= (1 - \alpha_i)a_{i+1}i - \sum_{j=i+2}^{n} (a_{ij} - \alpha a_{i+1,a_i+1,j})
\]

\[
= 2(1 - \alpha_i)a_{i+1}i - \sum_{j=i+1}^{n} (a_{ij} - \alpha a_{i+1,a_i+1,j})
\]

\[
= (u_i + 2a_{i+1}) + \alpha_i(r_i - 2a_{i+1}) > 0. \tag{17}
\]
Thus from (13) and (17) we easily obtain for \( 1 \leq \alpha_i < \alpha'_i \) (\( 1 \leq i < n \))

\[
d(\alpha)_i - l(\alpha)_i - \bar{u}(\alpha)_i
= (1 - l_i) - \alpha_i(p_i + q_i) - (u_i + 2a_{i+1}) - \alpha_i(r_i - 2a_{i+1})
= (1 - l_i - u_i - 2a_{i+1}) - \alpha_i(p_i + q_i + r_i - 2a_{i+1}) > 0.
\]

Therefore, \( A(\alpha) \) is a strictly diagonally dominant matrix, and thus the following equality holds:

\[
\rho(T(\alpha)) \leq \frac{\bar{u}(\alpha)_i}{d(\alpha)_i - l(\alpha)_i} < 1 \quad \text{for} \quad 1 \leq \alpha_i < \alpha'_i \quad (1 \leq i < n).
\]

Hence, an application of Lemma 2 yields \( \rho(T(\alpha)) < 1 \) for \( 1 \leq \alpha_i < \alpha'_i \) (\( 1 \leq i < n \)).

The behavior of the spectral radius of the proposed method as a function of \( \alpha_i = \alpha \) for the strictly diagonally dominant \( Z \)-matrix \( A \) is shown in Fig. 1.

The variation of the spectral radius of the proposed method is extremely small compared with that of the SOR method, as shown in Figure 1.

\[
\begin{array}{c}
\text{Spectral radius} \\
\hline
0.900000 & 0.800000 & 0.700000 & 0.600000 & 0.500000 & 0.400000 & 0.300000 & 0.200000 & 0.100000 & 0.000000 \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\alpha \ 3.35 \sim 4.34 \\
\omega \ 0.81 \sim 1.80 \\
\hline
\end{array}
\]

**FIG. 1.** The spectral radii of the proposed method and the SOR method for \( n = 10 \).
Moreover, our convergence curve is relatively flat for $\alpha > \alpha_{\text{opt}}$. However, it is extremely difficult to compute an optimal $\alpha_i$ directly from Theorem 4. Therefore we propose a practical technique for its determination.

To find $\alpha_i$, we dictate that the equality holds in (17):

$$(u_i + 2a_{i,i+1}) + \alpha_i(r_i - 2a_{i,i+1}) = 0, \quad 1 \leq i < n.$$ 

Solving this equation, we have

$$\alpha_i = \frac{u_i + 2a_{i,i+1}}{2a_{i,i+1} - r_i}, \quad 1 \leq i < n \quad (18)$$

4. NUMERICAL EXAMPLES AND CONCLUSION

We now test the validity of the determination (18). To do so, we consider the following matrix:

$$A = \begin{pmatrix}
1 & c_1 & c_2 & c_3 & c_1 & \cdots \\
c_3 & 1 & c_1 & c_2 & \cdots & c_1 \\
c_2 & c_3 & \ddots & \ddots & \ddots & c_3 \\
c_1 & \ddots & \ddots & 1 & c_1 & c_2 \\
c_3 & \ddots & c_2 & c_3 & 1 & c_1 \\
\cdots & c_3 & c_1 & c_2 & c_3 & 1
\end{pmatrix},$$

where $c_1 = -1/n$, $c_2 = -1/(n + 1)$, and $c_3 = -1/(n + 2)$. We set $b$ [see (1)] such that the solution is $x^T = (1, 2, \ldots, n)$. Let the convergence criterion be $\|x^{k+1} - x^k\|/\|x^{k+1}\| \leq 10^{-6}$. We show CPU times and the number of iterations in Table 1 for $n = 20, 30, 50, \text{ and } 100$. For comparison, we also show results for unpreconditioning (GS), the modified Gauss-Seidel method (MGS) [1], and the adaptive Gauss-Seidel method (AGS) [4].

The iteration number for the proposed method is larger than that for AGS [4], while the CPU time for the proposed method is smaller than that for AGS. An optimum parameter $\omega_{\text{opt}}$ of the SOR method was obtained by numerical computation. We also obtained the optimum parameter $\alpha_{\text{opt}}$ of the proposed method by replacing $\alpha$ with $\alpha_i$ ($i = 1, 2, \ldots, n - 1$) by numerical computation.
### TABLE 1

**Iteration Numbers and CPU Times for a Z-Matrix**

<table>
<thead>
<tr>
<th>Proposed method</th>
<th>Optimum</th>
<th>Determine</th>
<th>GS</th>
<th>MGS</th>
<th>AGS</th>
<th>SOR$_{opt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No. of iters.</td>
<td>Time (s)</td>
<td>No. of iters.</td>
<td>Time (s)</td>
<td>No. of iters.</td>
<td>Time (s)</td>
</tr>
<tr>
<td></td>
<td>$n$</td>
<td>$m$</td>
<td>$\alpha_{opt}$</td>
<td>$\alpha_{opt}$</td>
<td>$\omega_{opt}$</td>
<td>$\omega_{opt}$</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>19</td>
<td>10.4</td>
<td>0.01</td>
<td>65</td>
<td>0.02</td>
</tr>
<tr>
<td>30</td>
<td>23</td>
<td>17.4</td>
<td>0.01</td>
<td>48</td>
<td>0.02</td>
<td>93</td>
</tr>
<tr>
<td>50</td>
<td>28</td>
<td>32.3</td>
<td>0.04</td>
<td>80</td>
<td>0.10</td>
<td>146</td>
</tr>
<tr>
<td>100</td>
<td>38</td>
<td>72.9</td>
<td>0.22</td>
<td>156</td>
<td>0.62</td>
<td>269</td>
</tr>
</tbody>
</table>

### TABLE 2

**Model Problem**

<table>
<thead>
<tr>
<th>Proposed method</th>
<th>Optimum</th>
<th>Determine</th>
<th>GS</th>
<th>MGS</th>
<th>AGS</th>
<th>SOR$_{opt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No. of iters.</td>
<td>Time (s)</td>
<td>No. of iters.</td>
<td>Time (s)</td>
<td>No. of iters.</td>
<td>Time (s)</td>
</tr>
<tr>
<td></td>
<td>$m$</td>
<td>$n$</td>
<td>$\alpha_{opt}$</td>
<td>$\alpha_{opt}$</td>
<td>$\omega_{opt}$</td>
<td>$\omega_{opt}$</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>10</td>
<td>2.65</td>
<td>0.06</td>
<td>110</td>
<td>0.97</td>
</tr>
<tr>
<td>15</td>
<td>26</td>
<td>15</td>
<td>3.0</td>
<td>0.61</td>
<td>230</td>
<td>4.85</td>
</tr>
<tr>
<td>20</td>
<td>34</td>
<td>20</td>
<td>3.2</td>
<td>2.58</td>
<td>385</td>
<td>29.24</td>
</tr>
</tbody>
</table>
Finally, we consider a model problem [5, p. 202]. We use a standard central-difference formula and a uniform mesh with length $h = 1/m$. Table 2 shows CPU times and the number of iterations for the model problem. We adopt the theoretical value

$$\omega_{\text{opt}} = \frac{2}{1 + \sin(\pi/m)}$$

for the SOR method.

In this paper, we have proposed a new algorithm based on the Gauss-Seidel method. As a result we have succeeded in improving the convergence of this method. We have shown that the spectral radius of the proposed method with $\omega_{\text{opt}}$ is smaller than that of the SOR method.

REFERENCES